Extensions

The exercises on the next three pages extend the basic problem. These might be assigned as homework exercises. In addition, an alternate Pizza π scenario, with extensions, is provided. This could be used with Algebra 2 or higher level classes in which the solution to the basic example might seem obvious. In order to develop a deeper understanding of real-life applications of mathematics, students should be allowed to discuss these extensions, as well as other sorts of situations and the different constraints they would entail.

Extension 1: Changing the Minimum Wage

Suppose the minimum wage is increased to \$5.25 per hour. How does this change the formulation of the problem? Is the optimal solution the same? (Recall that the pay rate between the hours of 8 p.m. and midnight is \$7 per hour.)

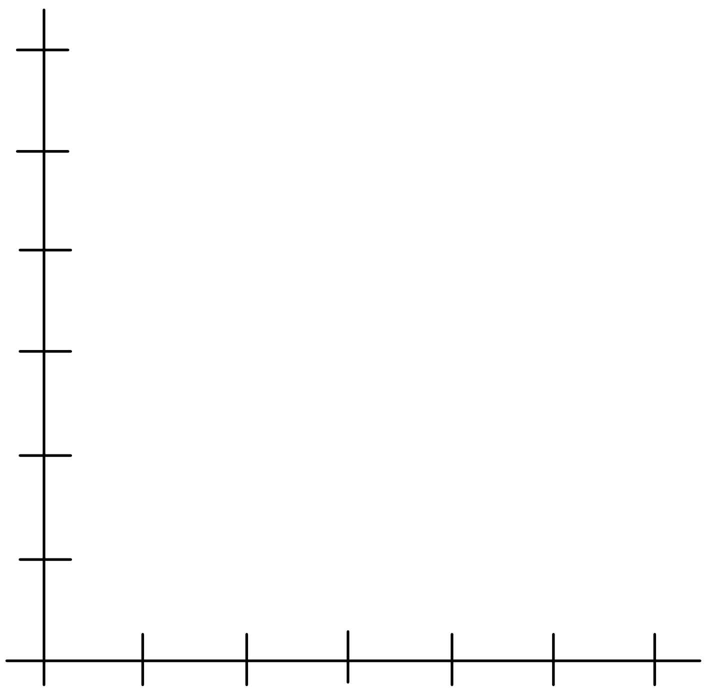
Solution: The formulation of the problem involves identifying the decision variables and constraints and constructing the objective function.

Decision Variables:

Constraints:

Objective Function:

To find the optimal solution, first locate the feasible region by graphing all of the constraints on the axes below:



Now, identify the corner points of the feasible region and test those points in the objective function:

What is the optimal solution?

Extension 2: Good Help is Hard to Find

Toni Pepperoni, the manager, has advertised for workers. Only 7 people have applied for jobs on the day shift and 12 on the evening shift. How does this change the problem formulation? Is the optimal solution the same?

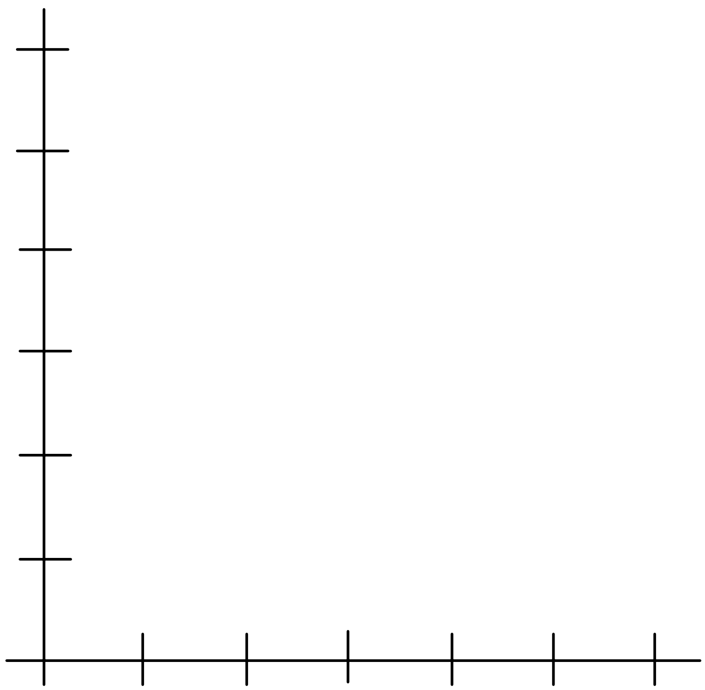
Solution: The formulation of the problem involves identifying the decision variables and constraints and constructing the objective function.

Decision Variables:

Constraints:

Objective Function:

To find the optimal solution, first locate the feasible region by graphing all of the constraints on the axes below:



Now, identify the corner points of the feasible region and test those points in the objective function:

What is the optimal solution?

Extension 3: Decreasing the Shift Pay Differential

Ms. Pepperoni recently met with the district manager of Pizza π, Mr. Italia. He suggested that if the pay on the day shift were increased to $6 per hour, Ms. Pepperoni would have no trouble hiring enough workers for the day shift. Reformulate the problem making this change. What is the optimal solution?

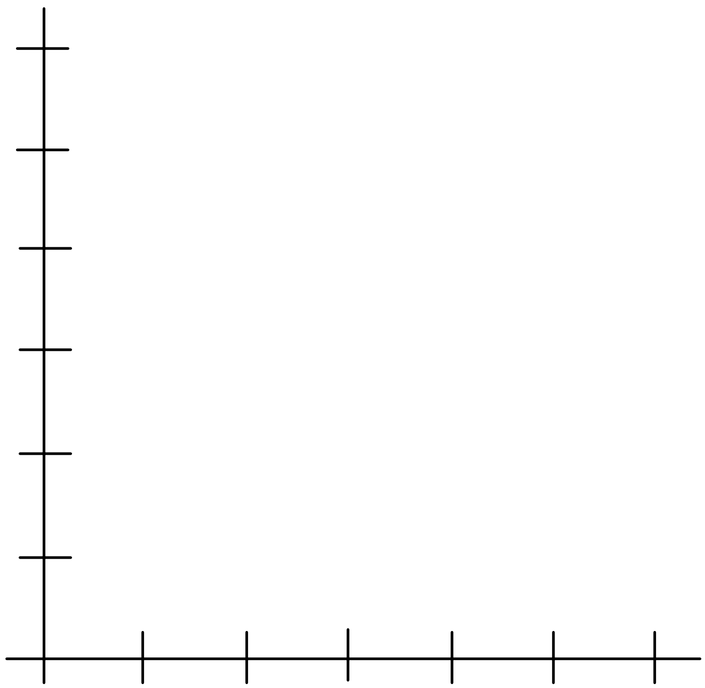
Solution: The formulation of the problem involves identifying the decision variables and constraints and constructing the objective function.

Decision Variables:

Constraints:

Objective Function:

To find the optimal solution, first locate the feasible region by graphing all of the constraints on the axes below:



Now, identify the corner points of the feasible region and test those points in the objective function:

What is the optimal solution?

Homework Problems:

1. Pizza π turns to part-time help.

Toni is considering hiring part-timers to work just the four hours between 4 p.m. and 8 p.m.

a) Create a new decision variable and rewrite the formulation for Pizza π.

b) You should be able to find the optimal solution using just logic. How much money would Toni save by using these part-timers?

c) Usually part-timers are paid even less than full-timers per hour. However, Toni is concerned about attracting good part-time help. If Toni offered the part-timers $8 an hour, would she still come out ahead?

d) Pizza π is considering expanding its operation to 24 hours a day. Toni has calculated the number of workers she will need in time-blocks of 2 hours. She is considering allowing a shift to start at any one of the two-hour blocks. She will hire both full-time workers (8 hour shifts) and half-time workers (4 hour shifts). How many decision variables and how many constraints will she have? How complex would the problem become if you were staffing a 1-800 hot-line 24 hours a day, 7 days a week and grouping data in 15 minute time blocks.

Teacher notes:

a) Let P = number of part-timers between 4 p.m. and 8 p.m., D = the number of day shift workers, and E = the number of evening shift workers.

Minimize \$40 D + 48 E + \$20 P

$D \geq 6$

$E \geq 8$

$D + E + P \geq 16$, $D \geq 0$, $E \geq 0$ and $P \geq 0$

b) The optimal solution is to set D and E to their minimums and fill in with part-timers.
D = 6, E = 8 and P =2, the Total Daily Wages = 40(6) + 48 (8) + 20 (2) =\$664
as compared to \$704 when only full-timers were hired. This is a net savings of \$40.

c) A salary of \$8, would mean each four-hour shift would cost \$32 instead of \$20. The total cost would now be \$688, which is still less than \$704. In fact, the use of part-timers would save money as long as the part-time hourly salary were less than \$10. At \$10 an hour, a four-hour part-time shift would match the salary for an entire 8-hour day shift.

d) There are 12 two-hour time blocks in a day. That means the problem would have 12 alternative decision variables for full-time help and 12 for part-time for a total of 24 decision variables. There would also be 12 constraints. There are 168 hours in a week, and four times as many 15-minute time blocks.

2. Day of Week Schedules: Dr. F. Nightingale of Bull Run Hospital

Dr. F. Nightingale is in charge of night shift of the emergency room of Bull Run Hospital. Based on historical data, she estimates that she needs the following number of nurses on duty on each day of the week.

Monday 8	Tuesday 4	Wednesday 6	Thursday 5
Friday 7	Saturday 9	Sunday 8	

The department currently uses only two 5-day shift schedules, A) Monday through Friday or B) Wednesday through Sunday. Obviously, the second shift schedule is less attractive and nurses on that schedule are paid 10% more. The pay for schedule type A is $160 per day and for schedule B is $176 per day.

a) Formulate the problem as a linear programming model. Identify any constraints that are obviously unnecessary (i.e. redundant) for solving this problem.

b) Find the optimal schedule. What is wasteful about this schedule?

c) Dr. Nightingale is considering changing schedule B to a shift that works from Thursday through Monday. What is the new optimal solution?

d) If the hospital could only use two schedules, what do you think would be the best possible pair of work schedules. Try to get the most from the overlapping days. (Assume that since staff did not have the weekend off, all nurses would be paid $176 per day.)

e) Reformulate the problem assuming that any five day work schedule is possible. Any work schedule that has nurses working on either Saturday or Sunday or both must pay the $176 salary for every one of the five days of work.

Teacher notes:

a) Let A = the number of nurses on schedule A
Let B = the number of nurses on schedule B

Minimize 800 A + 880 B subject to the following constraints:

$A \geq 8$, $A \geq 4$, $A + B \geq 6$, $A + B \geq 5$, $A + B \geq 7$, $B \geq 9$, $B \geq 8$, $A \geq 0$, and $B \geq 0$.

Only nurses on schedule A are on duty on Monday and Tuesday. Both sets of nurses are on duty on Wednesday, Thursday and Friday. Only schedule B nurses are on duty on Saturday and Sunday.

The second constraint is not needed since the first constraint already requires that A be greater than 8. The third and fourth constraints are also redundant since the fifth constraint requires that A + B must be greater than 7. The last constraint is also not needed since the next to last constraint, Saturday, requires that B be greater than 9. In reality none of the A + B constraints are needed with this formulation since the minimum value of A or B is greater than the minimum total required.

b) The optimal solution is A = 8 and B = 9 for a total cost of $14,320. On Wednesday, Thursday and Friday there will be 17 nurses on duty which is far more than the hospital needs.

c) By starting on Thursday and going to Monday, the schedules now overlap on Monday, a busy day. The new optimal is A= 6 and B = 9 for a total cost of $12,720.

d) The three busiest days are Saturday, Sunday and Monday. the best pair of schedules would overlap on every one of these three days. Therefore, one schedule should run from Saturday through Wednesday and the other schedule should run from Thursday through Monday. The Saturday start should have 6 nurses and the Thursday start should have 5 for a total cost of only $9,680.

e) There are seven different possible schedules and five sets of nurses could be working on any given day:

A = Number of nurses who from Monday through Friday
B = Number of nurses who from Tuesday through Saturday
C = Number of nurses who from Wednesday through Sunday
D = Number of nurses who from Thursday through Monday
E = Number of nurses who from Friday through Tuesday
F = Number of nurses who from Saturday through Wednesday
G = Number of nurses who from Sunday through Thursday

In writing the constraints, students often confuse the number of nurses starting on a particular day of the week with the total working that day. Any nurse whose shift starts on Monday, Thursday, Friday, Saturday, or Sunday will be on duty on Monday. Minimize:

$$800\,A + 880\,B + 880\,C + 880\,D + 880\,E + 880\,F + 880\,G, \text{ subject to:}$$

On duty Monday:	$A + D + E + F + G$	≥ 8
On duty Tuesday:	$A + B + E + F + G$	≥ 4
On duty Wednesday:	$A + B + C + F + G$	≥ 6
On duty Thursday:	$A + B + C + D + G$	≥ 5
On duty Friday:	$A + B + C + D + E$	≥ 7
On duty Saturday:	$B + C + D + E + F$	≥ 9
On duty Sunday:	$C + D + E + F + G$	≥ 8

3. Linking Time Periods

Most manpower planning models look at multiple time periods (months, quarters or years) and require equations that link one time period with the next time period. A subscript is used to write a decision variable.

Let: W_t = the number of workers on staff at the beginning of month t
H_t = the number of workers hired during month t
L_t = the number of workers laid off during month t
Q_t = the number of workers who quit during month t.

The decision variables H_t and L_t are under management control; whereas Q_t is not. The number of workers on staff at the end of a period is a function of the staff at the beginning of the period, key management decisions on hiring and layoffs, and worker personal decisions about quitting.

A standard equation would look like:

$$W_{t+1} = W_t + H_t - L_t - Q_t.$$

With regard to military planning, a separate set of variables is needed for each rank. Thus one new decision variable would be:

P_t = the number of people promoted during the time period.

a. Write a similar equation linking the number of sophomores in your school at the beginning of this year with the number of juniors in your school at the beginning of next year.
b. Write a similar equation linking the population of your home town on January 1 of this year with the population on January 1 of next year.

In each case, be sure to clearly define the variables you use.

Extensions to Base Problem:

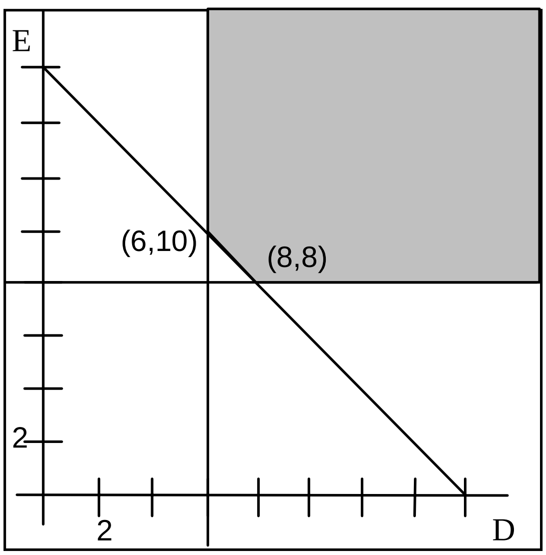
1 The decision variables are the same:

D = # of day shift workers,
E = # of evening shift workers

The system of constraints is also unchanged: $D \geq 6$, $E \geq 8$, and $D+E \geq 16$.

Changing the minimum wage does affect the objective function:
Total daily wages, W, is given by:

$$W = (5.25\cdot 8)D + [(4)(5.25)+(4)(7)]E$$
$$= 42D + 49E.$$



Since the system of constraints is the same, we have the same two corner points: (6,10) and (8,8).

For (6,10), W = 42(6)+49(10) = $742.
For (8,8), W = 42(8)+49(8) = $728.

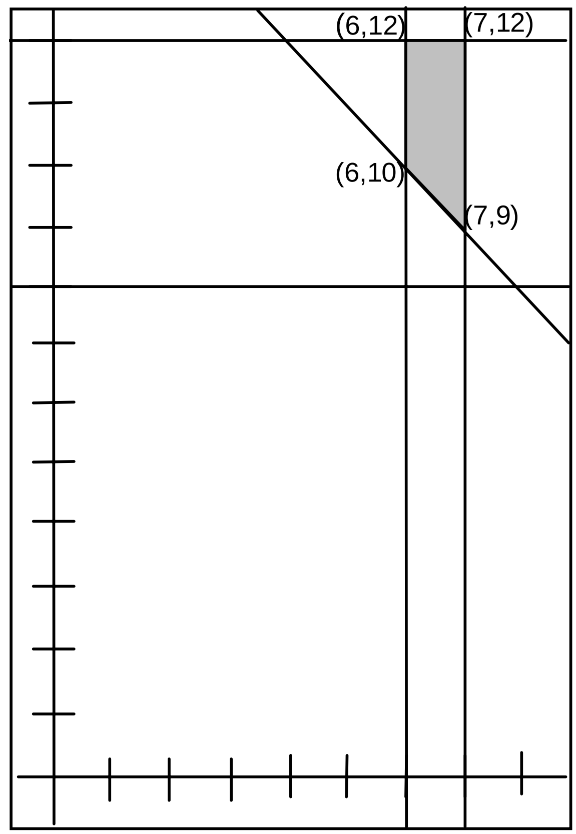
(8,8), or 8 workers on each of the two shifts remains the optimal solution.

2 The decision variables are the same:

D = # of day shift workers,
E = # of evening shift workers

Two of the constraints have been changed: $6 \leq D \leq 7$; $8 \leq E \leq 12$; the other constraint ($D + E \geq 16$) remains the same.

Using the same pay scale as in the base problem, we have the same objective function: 40D + 48E = total daily wages.



For (6,12), W = 40(6)+48(12) = $816
For (7,12), W = 40(7)+48(12) = $856
For (6,10), W = 40(6)+48(10) = $720
For (7,9), W = 40(7)+48(9) = $712

(7,9), or 7 day shift and 9 evening shift workers is now the optimal solution.

3 The decision variables and the system of constraints are the same as in the base problem. Changing the pay rate for the day shift changes the objective function:

$$W = (8 \cdot 6)D + [(4)(6)+(4)(7)]E$$
$$W = 48D + 52E$$

The graph of the feasible region is the same as the graph in the first extension.

For (6,10), 48(6)+52(10) = $808.
For (8,8), 48(8)+52(8) = $800.

(8,8), or 8 workers on each shift is again the optimal solution.