Extensions

Four problem situations, which extend the ideas in the Arm-and-a-Leg problem, follow. In extension 1, students manipulate some of the formulas underlying a single-server model. In extensions 2-4, multi-server models are introduced. In the second and third extensions, students will use the table immediately following extension 2 to determine appropriate values of L. In extension 4, students manipulate the two-server formulas. The formulas for systems which use more than two servers are quite difficult and beyond the scope of this module.

In extensions 2 and 3, linear interpolation, the insertion of an approximate value between two values in a table, must be used. In extension 4, notice that for a multi-server model a may be greater than h, but x must still be less than 1. This leads quite naturally to a different defining equation for x. In a multi-server system with n servers,

$$x = \frac{a}{nh}.$$

Extension 1: Manipulate Equations

1. If $L = \dfrac{x}{1-x}$ and $x = \dfrac{a}{h}$, replace x with $\dfrac{a}{h}$. Simplify the resulting fraction. You should now have an expression for L in terms of a and h.

2. Use the previous result and the formula $L = aW$ to find W in terms of a and h and simplify the resulting expression.

3. W represents the average total time in the system, including the average time waiting in line, W_q, and the average time being served, $1/h$.

   a. Express W in terms of W_q and h.

   b. Rewrite the equation for W and substitute to find W_q in terms of a and h.

4. a. Using $a = 18$ and $h = 20$, find W_q for the Arm-and-a-Leg problem.

   b. What does this value of W_q mean?

Extension 2: Add Servers – Evaluate Impact Using Tables

One of the ways to improve service at Arm-and-a-Leg Tickets that Dr. Cue might have suggested to Mr. I. M. Boss is the addition of one or more servers at his busiest locations. This creates a multi-server queueing system. In this case, the mathematical formulas are much more complex, but tables of values for L can be used to analyze such systems. The key quantity that must be computed is f, the fraction of time each server is busy, on average.

In a single server system, if customers arrive at the average rate of a per hour and are served at the average rate of h per hour, then a/h represents the proportion of time the server is busy, on average.

1. Assume that $a = 18$ and $h = 20$. What is the average proportion of time the server is busy?

Another way to look at the ratio a/h is that it represents the fraction of the total available service time that is actually being used.

2. Assume that there are n servers in the system. In terms of a, h, and n, represent f, the fraction of each server's available time that is actually being used.

Suppose that Mr. I. M. Boss hires a second server at each of his busiest outlets.

3. Using the same values of a and h, on average, what fraction of the time will each of the two servers be busy?

The accompanying table (see next page) contains values of L for various values of n and f.

4. If $n = 2$ and $f = 0.4$, what is the value of L given in the table?

5. If $n = 2$ and $f = 0.5$, what is the value of L given in the table?

6. What is an appropriate value of L for $n = 2$ and the value of f you computed in question 3.

7. What does this value of L mean in the Arm-and-a-Leg problem?

8. Recall that $W = L/a$. Compute W for the Arm-and-a-Leg problem using two servers.

9. What does this value of W mean?

10. How do the values of L and W in the two-server example compare to the corresponding values from the original (single-server) problem?

L

***n*:**	**1**	**2**	**3**	**4**	**5**	**6**
f						
0.10	0.111	0.202	0.300	0.400	0.500	0.600
0.20	0.250	0.417	0.606	0.802	1.001	1.200
0.30	0.429	0.659	0.930	1.216	1.509	1.805
0.40	0.667	0.952	1.294	1.661	2.040	2.427
0.50	1.000	1.333	1.737	2.174	2.630	3.099
0.55	1.222	1.577	2.008	2.477	2.969	3.475
0.60	1.500	1.875	2.332	2.831	3.354	3.895
0.65	1.857	2.251	2.732	3.258	3.812	4.385
0.70	2.333	2.745	3.249	3.800	4.382	4.984
0.75	3.000	3.429	3.953	4.528	5.135	5.765
0.80	4.000	4.444	4.989	5.586	6.216	6.871
0.85	5.667	6.126	6.689	7.306	7.959	8.636
0.90	9.000	9.474	10.054	10.690	11.362	12.061
0.95	19.000	19.487	20.083	20.737	21.428	22.146
0.98	49.000	49.495	50.100	50.764	51.466	52.194
0.99	99.000	99.498	100.106	100.773	101.478	102.210

L = the average number of customers in the system

f = fraction of time the server(s) are busy

n = number of servers

Extension 3: Determine Number of Servers to Add

Recall from Extension 2 that in a multi-server queueing system having *n* servers with an arrival rate of *a* customers per hour and a customer service rate of *h*

$$f = \frac{a}{nh}$$

customers per hour for each server, the fraction of time that each server is busy is given by:

At the Eastworst Airlines ticket counter at Metropolis International Airport, the number of customers arriving averages 76 per hour. The average time necessary for ticket purchase and baggage checking at the counter is 3 minutes for each customer.

1. On average, how many customers can each ticket agent handle per hour?

2. What is the fewest number of ticket agents needed to keep the line at the Eastworst ticket counter from growing indefinitely?

3. If the Eastworst queueing system uses the fewest number of ticket agents from question 2, what fraction of the time will each ticket agent be busy?

4. Using the table on p. 16, on average, how many people will be in the Eastworst queueing system?

5. The formula $W = L/a$ also applies to a multi-server system. Using the Eastworst values of L and a, what is the average time an Eastworst customer will spend at the ticket counter, waiting in line and being served?

The management at Eastworst Airlines has received numerous complaints about the time it takes to be served at their ticket counter at Metropolis International Airport and is considering adding one or two additional ticket agents to the system.

6. For the values of n that are 1 and 2 more than your value of n in question #2, calculate the values of f, L, and W.

7. If you were the manager of the ticket counter at Eastworst Airlines, would you add one ticket agent or two? Explain your reasoning.

Extension 4: Two-Server Formula

To address Arm & a Leg's customer satisfaction, Dr. Cue recommended that Mr. Boss increase the average number of customers serviced per hour. One way to do this is to hire a faster server (one with a larger h). Another way is to add a second comparable server to lighten the single-server workload and speed up service.

Assuming that the two servers have identical service rates, h, here are the formulas for a two-server single-line queue system in terms of a and h:

$$L_q = \frac{a^3}{4h^3 - ha^2},$$

$$L = \frac{4ah}{4h^2 - a^2},$$

$$W = \frac{L}{a},$$

where L_q is the average number of customers in the line, L is the average number of customers in the system, W is the average total time waiting in the system, a is the average customer arrival rate, and h is the average customer service rate.

L is related to L_q by the equation:

$$L = L_q + \frac{a}{h}$$

The formulas, written in this form for L_q and L, allow us to find the values of L_q and L, given the arrival and service rates, a and h. The formula relating L and W applies to all types of queueing models and thus allows us to manipulate the formulas for L and W.

1. Write the formula for W in terms of a and h. (Hint: substitute L from above into the formula $L = aW$).

2. Now verify that:

$$L = L_q + \frac{a}{h}$$

(Hint: substitute in L and L_q, and verify in terms of a and h.)

3. Suppose Mr. Boss hires another server who also has a customer service rate of h = 20 customers per hour. Complete the following table given the values of a and h.

a customer/ hr	h customer/ hr	L_q customers in line	L Total customers	W wait time (hr)	W wait time (min)
28	20	1.35	2.75	0.10	5.9
30	20				
32	20				
35	20				
38	20				
39	20				

Homework Problems:

1.a.) Read the case studies and classify each as a single-server, multiple single-servers, parallel servers, or servers-in-sequence model.

b.) Give a situation where you have encountered each type of server model.

2.) At an outdoor concert, there are two portable toilets designated for females and one for males. Women wait in one line to use one of the two portable toilets. Assume that the average customer arrival rate at both the male and female portable toilets is 30 per hour, but the average service time for females is three minutes and for males is one minute. Are the portable toilets distributed fairly? Use queueing theory to support your answer.

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3.) At a local library, one clerk is checking out books. On average, 40 people per hour arrive at the counter to check out books. It takes an average of one minute to service a person.

a.) On average, how many people are in line and being serviced?

b.) On average, how many people are in line?

c.) On average, how much time will a person spend in line and at the counter checking out books?

d.) On average, how much time will a person spend in line?

4) Your school is hosting the Homecoming Game in two weeks. Your job is to set up a reasonable queueing model to manage ticket taking and admission to the game.

a.) Using your school as a model, estimate each of the following:

The number of people expected at the game (based on your school population),

The average time it takes to take a ticket and admit someone,

Calculate the average number of customers served by one server (ticket taker) per minute and per hour.

The number of totally separate entrances (e.g. opposite sides of the arena) where ticket takers will be stationed.

b.) Before you can develop an appropriate queueing model, you will need to determine the average arrival rate per minute.

During how long a time period will the vast majority of fans arrive at the game?

Use this time length and the number of people listed in part a) to determine the average arrival rate of fans to the ticket-taking gate.

If you assumed that there was more than one entrance, you will need to determine an arrival rate for each gate.

c.) Discuss the type of queueing model(s) you could use to plan the number of ticket takers at each entrance. (When separate lines are formed far apart so that people can not jump back and forth to the shorter line, a separate queuing model is used for each location.)

In order to decide how many ticket takers to have, you will want to establish a customer service standard. Specify this standard as the average time, W, customers will spend in line before being admitted.

d.) Every mathematical model used to analyze a real-world situation involves making simplifying assumptions. Discuss assumptions you made in order to determine the average arrival rate.

e.) Find each of the following:

The minimum number of ticket takers required at each gate so that the total service rate exceeds the arrival rate. Why is this number important?

The traffic intensity, x.

The average number of customers in the system, L, at each entrance.

The average waiting time for each customer, W.

The fraction of each server's available time being used per hour, f.

f.) Earlier you specified a service standard for W. Convert that standard into a value for L. Use the tables to determine the minimum number of ticket agents that will be needed to achieve this standard. Create a simple table that specifies the value of f as the number of servers is increased above the absolute minimum.

g.) Assume that all of the fans arrive during the hour before the game but the average arrival rate is not constant. Break this hour into 4 fifteen-minute intervals. What percentage of the fans do you estimate will arrive during each of these four time-periods. Use these percentages to determine an average arrival rate for each 15-minute interval. Repeat steps e) and f) for each period.

h.) Write your conclusions in a two-page form suitable to be used as a proposal to your athletic director for management of ticket sales. Support your conclusions.

L

n:	**1**	**2**	**3**	**4**	**5**	**6**
f						
0.10	0.111	0.202	0.300	0.400	0.500	0.600
0.20	0.250	0.417	0.606	0.802	1.001	1.200
0.30	0.429	0.659	0.930	1.216	1.509	1.805
0.40	0.667	0.952	1.294	1.661	2.040	2.427
0.50	1.000	1.333	1.737	2.174	2.630	3.099
0.55	1.222	1.577	2.008	2.477	2.969	3.475
0.60	1.500	1.875	2.332	2.831	3.354	3.895
0.65	1.857	2.251	2.732	3.258	3.812	4.385
0.70	2.333	2.745	3.249	3.800	4.382	4.984
0.75	3.000	3.429	3.953	4.528	5.135	5.765
0.80	4.000	4.444	4.989	5.586	6.216	6.871
0.85	5.667	6.126	6.689	7.306	7.959	8.636
0.90	9.000	9.474	10.054	10.690	11.362	12.061
0.95	19.000	19.487	20.083	20.737	21.428	22.146
0.98	49.000	49.495	50.100	50.764	51.466	52.194
0.99	99.000	99.498	100.106	100.773	101.478	102.210

L = the average number of customers in the system

f = fraction of time the server(s) are busy

n = number of servers

Solution Key

Extension 1

1.

$$\frac{a}{h-a}$$

2.

$$W = \frac{1}{h-a}$$

3a.

$$W = W_q + \frac{1}{h}$$

b.

$$W_q = \frac{a}{h(h-a)}$$

4a.

$$W_q = \frac{9}{20} = 0.45 \text{ hours}$$

b. In our case, customers spend 9/20 hours or 27 minutes waiting in line.

Extension 2

1.

$$x = \frac{18}{20} = 0.9$$

2.

$$f = \frac{a}{nh}$$

3.

$$f = 0.45 \text{ or } 45\%$$

4. 0.952

5. 1.333

6. Using linear interpolation,

$$0.5(1.333 + 0.952) = 1.1425 \approx 1.143$$

7. The average number of customers in the system if 2 servers are used.

8.

$$W = \frac{L}{a} = \frac{1.143}{18} = 0.0635 \text{ hrs} = 3.81 \text{ min}$$

9. The average time a customer waits in the 2 server system.

10. L and W are much smaller.

Extension 3

1.

$$L = \frac{60 \text{ min}}{3 \text{ min / customer}} = 20 \text{ customers}$$

2. The smallest n such that

$$f = \frac{a}{nh} = \frac{76}{20n} < 1; \text{ therefore } n = 4$$

3.

$$f = \frac{a}{nh} = \frac{76}{4(20)} = \frac{76}{80} = \frac{19}{20} = 0.95 \text{ or } 95\%$$

4. L = 20.737

5. W = 0.273 hours or 16.38 minutes

6.

n	f	L	W
5	0.76	5.351	0.070 hrs or 4.20 min
6	0.63	4.189	0.055 hrs or 3.31 min

7. Add one more server, because the advantage gained by adding two, compared to adding only one, probably does not justify the cost of the extra server.

Extension 4

1.

$$W = \frac{4h}{4h^2 - a^2}$$

2.

Show that $\frac{4ah}{4h^2 - a^2} - \frac{a^3}{4h^3 - ha^2} = \frac{a}{h}$

$$\left(\frac{h}{h}\right)\left(\frac{4ah}{4h^2 - a^2}\right) - \frac{a^3}{h(4h^2 - a^2)} = \frac{4ah^2 - a^3}{h(4h^2 - a^2)} = \frac{a(4h^2 - a^2)}{h(4h^2 - a^2)} = \frac{a}{h}$$

3.

a customer/ hr	*h* customer/h r	L_q customers in line	*L* Total customers	*W* Total wait time in hours	*Min* Total wait time in minutes
28	20	1.35	2.75	0.10	5.9
30	20	1.93	3.43	0.11	6.8
32	20	2.84	4.44	0.14	8.3
35	20	5.72	7.47	0.21	12.8
38	20	17.59	19.49	0.51	30.8
39	20	37.54	39.49	1.01	60.8

Homework Solutions

1a) The Toilets Down Under-- Multiple parallel servers of different types
Department of Motor Vehicles--Servers in a Series
The Coal Unloader--Single Server
L. L. Bean--Parallel Server

1b) Answers will vary depending on students' experiences.

2) The toilets are not fairly distributed. Males will be in the system a total of two minutes with a one-minute wait. Females will be in the system approximately 6.86 minutes with a waiting time in line of 3.86 minutes.

3a) On average, two people will be in the system (both in line and being serviced).

3b) On average, 1.333 people will be in line.

3c) A person will spend a total of three minutes in line and checking out.

3d) A person will spend two minutes in line.

4a) In gathering the data, the teacher may facilitate this portion of the homework by asking the Director of Athletics for this information. There may be different scenarios given to different students.

To determine this statistic, students can perform a simple experiment that simulates a group of people waiting in line with tickets to enter a room. They could time how long it takes to admit someone.

To calculate the average number of customers served by one server per minute and per hour invert the service time and scale the number appropriately.

The number of totally separate entrances is important because the students will need to divide the total arrival rate by the number of entrances to find the arrival rate per entrance.

4b) The total attendance statistic will need to be divided by the time period chosen and the number of entrances to determine the arrival rate per hour or minute.

4c) Each entrance should be treated as a separate entity that is either a single or multiple server system with just one line. Even if there are two ticket takers at an entrance and it looks as if there are two lines, as long as people can jockey between the two lines, the system behaves as if it were a single queue in front of multiple servers.

4d) **Assumptions**
The average arrival rate of attendees during the time period specified is approximately a constant average rate of random arrivals.
Arrivals of people and admitting them will be viewed as individual cases even if some people arrive in groups of two or more.
Arriving customers will be divided evenly amongst the gates.
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4e) The minimum number of servers, n, is the smallest number larger than the ratio of (a/h). This insures that the total service rate exceeds the total arrival rate.

The average fraction of time servers are busy is just (a/nh)

Use this value of f and n to look up in the table to estimate the value of L. If *f* is midway between the table values you may want the students to extrapolate to find L. W is found using the standard formula

$$W = L/a.$$

4f) Use the standard set for W, to determine the target value of L. Once L has been specified, it becomes somewhat tricky to use the table. Remember each value of L in the table corresponds to a pair of *f* and n values. Students should determine for each value of n in their problem, the corresponding value of *f* and circle the resulting estimate for L. They should identify the first column in which L drops below the standard they have set.

4g) Students may want to assume that the arrival pattern is similar to the following:
60 – 45 minutes before game time 10% arrive
45 – 30 minutes before game time 25% arrive
30 – 15 minutes before game time 45% arrive
15 – 0 minutes before game time 20% arrive
These percentages are then used to calculate the average arrival rate per minute for each fifteen-minute interval.

We will be treating each time period as a separate independent entity. However, if in reality long lines exist at the end of one time period and carryover into the next, this assumption can produce a poor approximation. Advanced queueing models were developed in order to relax this independence assumption when queueing models were used to forecast delays in responding to police emergencies in NYC.

4h) The points to be considered in writing a rubric for the paragraph should be:

- The number of servers needed to work at the game
- The length of time each of the sellers need to work
- Customer satisfaction
- A justification for using your queueing model at the game
- Decreasing the stress on customers and sellers
- Decreasing the congestion of patrons at the ticket booth